The evolutions of spinning bodies moving in rotating black hole spacetimes

Zoltán Keresztes
Balázs Mikóczi

Department of Theoretical Physics, University of Szeged
Research Institute for Particle and Nuclear Physics, Wigner RCP

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- The work of B. M. was supported by the Janos Bolyai Research Scholarship of the Hungarian Academy of Sciences

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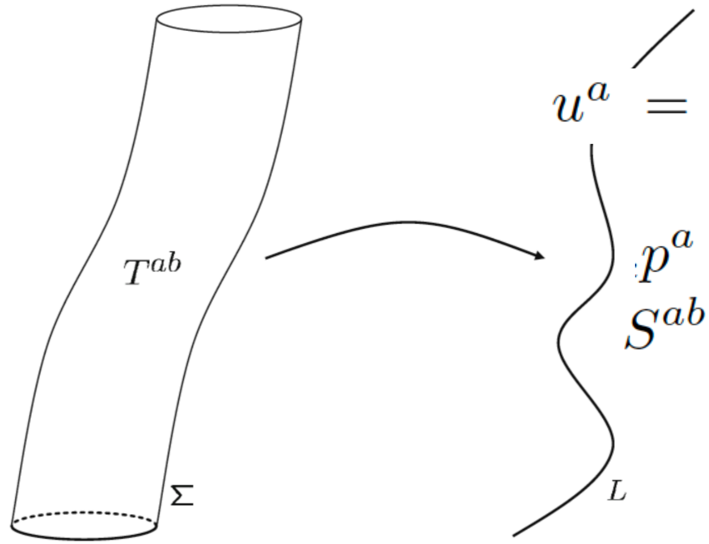
Mathisson-Papapetrou-Dixon Eqs.

Description of extended body with multipole moments:



$$u^a = dx^a/d\tau \qquad (u^a u_a = -1)$$

$$\frac{Dp^a}{d\tau} \equiv u^c \nabla_c p^a = F^a \; , \qquad F^a = -\frac{1}{2} R^a{}_{bcd} u^b S^{cd}$$

$$\frac{DS^{ab}}{d\tau} \equiv u^c \nabla_c S^{ab} = p^a u^b - u^a p^b \; ,$$

$$p^a = m u^a - u_b \frac{DS^{ab}}{d\tau}$$

Kinematical mass: $m = -u_a p^a$

Dynamical mass: $M = \sqrt{-p^a p_a}$

M. Mathisson, Acta. Phys. Polon. **6**, 163 (1937).
A. Papapetrou, Proc. Phys. Soc. **64**, 57 (1951).
W. Dixon, Nuovo Cim. **34**, 317 (1964).

Spin Supplementary Conditions (SSCs)

Frenkel-Mathisson-Pirani (FMP) SSC

J. Frenkel, Z. Phys. **37**, 243 (1926).
M. Mathisson, Acta Phys. Polon. **6**, 163 (1937).
F.A.E. Pirani, Acta Phys. Polon. **15**, 389 (1956).

$$u_a S^{ab} = 0$$

Constants:

O. Semerák, Mon. Not. Roy. Astron. Soc. **308**, 863 (1999).

$$\frac{D}{d\tau}\left(S_{ab}S^{ab}\right) = 0 \ ,$$

$$\frac{dm}{d\tau} = 0$$

Velocity-momentum relation:

L.F.O. Costa, G. Lukes-Gerakopoulos, O. Semerák, Phys. Rev. D **97**, 084023 (2018).

$$m u^a = p^a + \frac{2S^{ab}S_{bc}p^c}{S^{de}S_{de}} \ .$$

Tulczyjew-Dixon (TD) SSC

W.M. Tulczyjew, Acta Phys. Polon. **18**, 393 (1959).
W. Dixon, Nuovo Cim. **34**, 317 (1964).

$$p_a S^{ab} = 0$$

Constants:

O. Semerák, Mon. Not. Roy. Astron. Soc. **308**, 863 (1999).

$$\frac{D}{d\tau}\left(S_{ab}S^{ab}\right) = 0 \ ,$$

$$\frac{dM}{d\tau} = 0$$

Velocity-momentum relation:

K.P. Tod, F. de Felice, Il Nouvo Cimento **34**, 365 (1976).

$$u^b = \frac{m}{M^2}\left(p^b + \frac{2S^{ba}R_{aecd}p^e S^{cd}}{4M^2 + R_{aecd}S^{ae}S^{cd}}\right)$$

Spin vectors with SSCs

O. Semerák, Mon. Not. Roy. Astron. Soc. **308**, 863 (1999).

Frenkel-Mathisson-Pirani (FMP) SSC

Spin vector: $s^a = -\frac{1}{2}\eta^{abcd}u_b S_{cd}$

$$S^{ab} = \eta^{ab}{}_{cd}u^c s^d$$

Spin magnitude: $s_a s^a = \frac{1}{2}S_{cd}S^{cd}$

Orthogonality relations:

$$s_a S^{ab} = 0\,,\quad s_a u^a = 0\,,$$

$$s_a p^a = 0$$

Equation of motion:

$$\frac{Ds^a}{d\tau} = u^a a_b s^b$$

- The case of negligible acceleration was investigated in Ref.

D. Bini, A. Geralico, R.T. Jantzen, Phys. Rev. D **95**, 124022 (2017).

Tulczyjew-Dixon (TD) SSC

Spin vector: $S^a = -\frac{1}{2M}\eta^{abcd}p_b S_{cd}$

$$S^{ab} = \frac{1}{M}\eta^{ab}{}_{cd}p^c S^d$$

Spin magnitude: $S_a S^a = \frac{1}{2}S_{cd}S^{cd}$

Orthogonality relations:

$$S_a S^{ab} = 0\,,\quad p_a S^a = 0\,,$$

$$S_b u^b = 0$$

Equation of motion:

$$\frac{DS^a}{d\tau} = \frac{S^b F_b}{M^2}p^a$$

Kerr spacetime

R. P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).

Line element squared:

$$ds^2 = -\frac{\Delta - a^2 \sin^2\theta}{\Sigma} dt^2 - \frac{2a\mathcal{B}\sin^2\theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{\mathcal{A}}{\Sigma}\sin^2\theta d\phi^2$$

$$\Sigma = r^2 + a^2\cos^2\theta \ , \ \ \Delta = r^2 + a^2 - 2\mu r \ ,$$

$$\mathcal{B} = r^2 + a^2 - \Delta \ , \ \mathcal{A} = \left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2\theta \ .$$

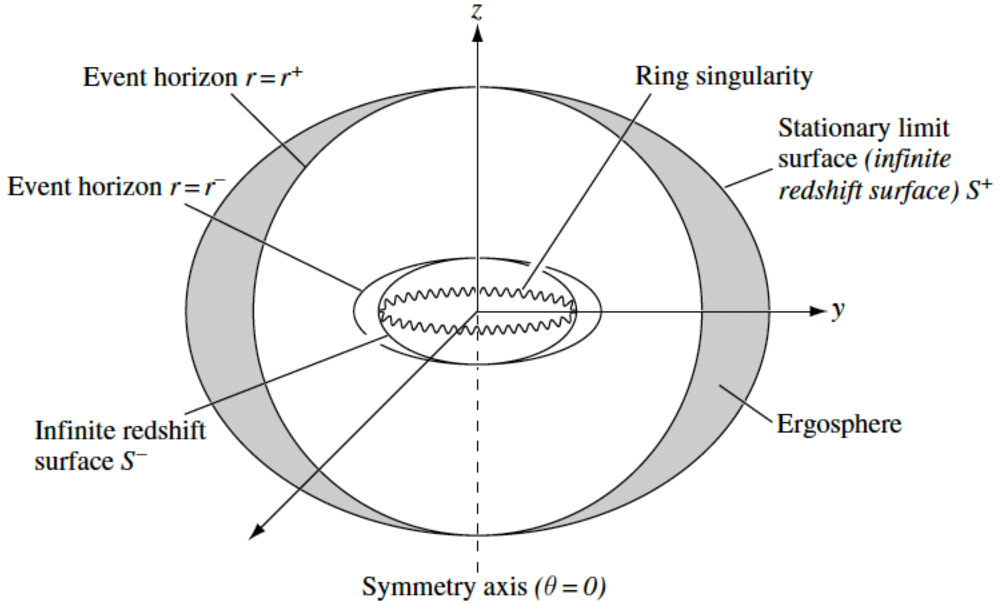
Stationary limit surfaces: $g_{tt} = 0$

Event horizons: $g^{rr} = 0$

Constants of motion:

$$E = -p_t - \frac{1}{2} S^{ab} \partial_a g_{bt} \ ,$$

$$J_z = p_\phi + \frac{1}{2} S^{ab} \partial_a g_{b\phi} \ .$$



Rotating Bardeen-like and Hayward-like spacetimes

B. Toshmatov, Z. Stuchlík, B. Ahmedov, Phys. Rev. D **95**, 084037 (2017).

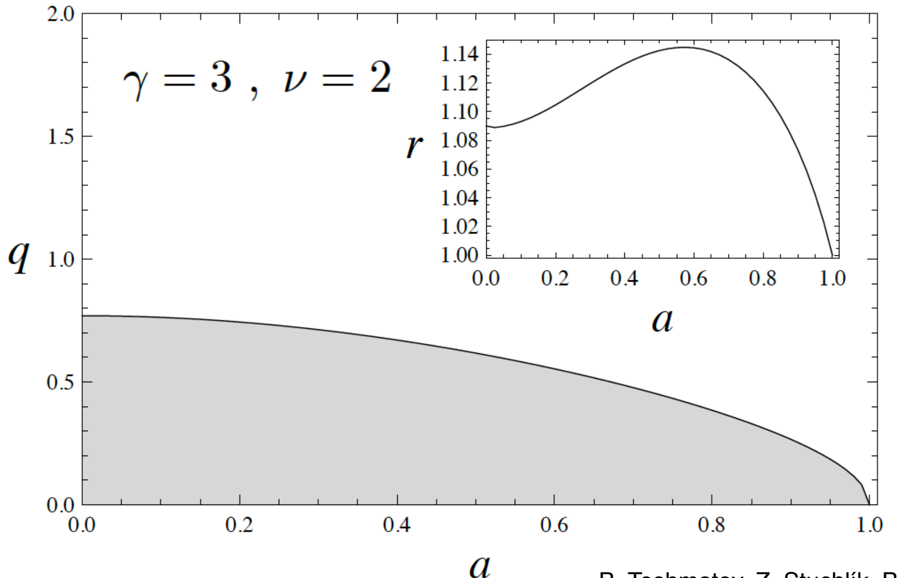
Line element squared:

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 - \frac{2a\mathcal{B}\sin^2\theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\mathcal{A}}{\Sigma}\sin^2\theta d\phi^2$$

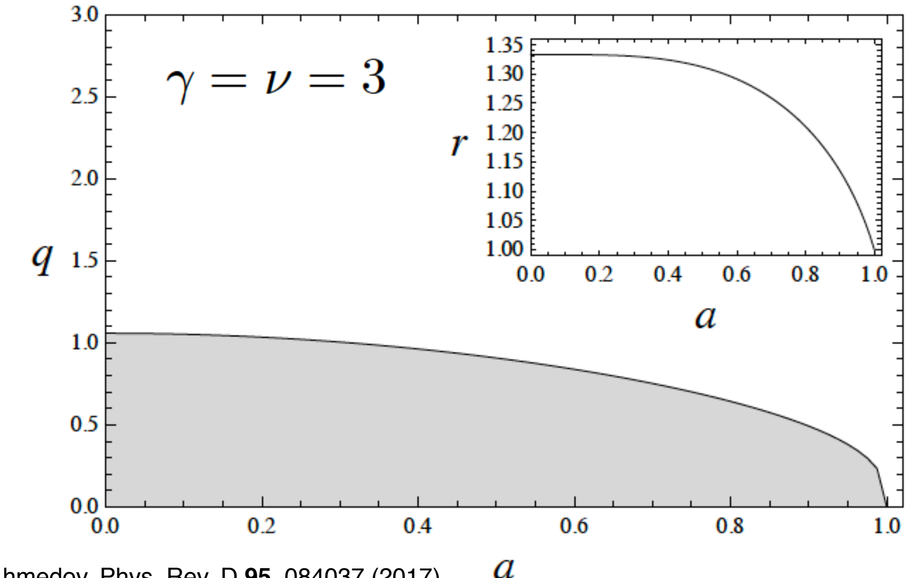
$$\Sigma = r^2 + a^2\cos^2\theta\ , \quad \mathcal{B} = r^2 + a^2 - \Delta\ , \quad \mathcal{A} = \left(r^2 + a^2\right)^2 - \Delta a^2\sin^2\theta\ ,$$

$$\Delta = r^2 + a^2 - 2\mu r\frac{r^\gamma}{\left(r^\nu + q_m^\nu\right)^{\gamma/\nu}}\ . \qquad \mu = \frac{q_m^3}{\sigma}\ , \quad q = \frac{q_m}{\mu}$$

Bardeen-like



Hayward-like



B. Toshmatov, Z. Stuchlík, B. Ahmedov, Phys. Rev. D **95**, 084037 (2017).

Comoving and zero 3-momentum frames

Comoving frame: $U^a = u^a$ — FMP or TD SSC

zero 3-momentum frame: $U^a = p^a/M$ — TD SSC

SO: $u_{(SO)} = \frac{1}{\sqrt{-g_{tt}}}\partial_t$

SO frame vectors:

$$e_0 = u_{(SO)}\ ,\ e_1 = \sqrt{\frac{\Delta}{\Sigma}}\partial_r\ ,\ e_2 = \frac{\partial_\theta}{\sqrt{\Sigma}}\ ,$$

$$e_3 = -\frac{1}{\sqrt{\Delta}}\left(\frac{a\mathcal{B}\sin\theta}{\Sigma\sqrt{-g_{tt}}}\partial_t - \frac{\sqrt{-g_{tt}}}{\sin\theta}\partial_\phi\right)\ .$$

$$s_a u^a = 0\ ,\ S_b u^b = 0\ , p_a S^a = 0$$

Boost transformation: D. Bini, A. Geralico, R.T. Jantzen, Phys. Rev. D **95**, 124022 (2017).

→ $E_0(e, U) \equiv U$

$E_\alpha(e, U) \quad (\alpha = \{1, 2, 3\})$

Related by a spatial rotation in U-frame: ↕ Rotation angle: Θ

ZAMO: $u_{(ZAMO)} = \sqrt{\frac{\mathcal{A}}{\Sigma\Delta}}\left(\partial_t + \frac{a\mathcal{B}}{\mathcal{A}}\partial_\phi\right)$

ZAMO frame vectors: $f_0\ ,\ f_\alpha$

Boost transformation:

→ $E_0(f, U) \equiv U$

$E_\alpha(f, U) \quad (\alpha = \{1, 2, 3\})$

Spin equations in comoving and zero 3-momentum frames

FMP SSC:

$U^a = u^a$

$$\frac{ds^\alpha}{d\tau} + \varepsilon^\alpha{}_{\beta\gamma}\Omega^\beta s^\gamma = 0 \ .$$

$$\Omega^\alpha = -\frac{1}{2}\varepsilon^{\alpha\beta\gamma}E_\beta \cdot \frac{DE_\gamma}{d\tau}$$

TD SSC:

$U^a = p^a/M$

$$\frac{dS^\alpha}{d\tau} + \varepsilon^\alpha{}_{\beta\gamma}\Omega^\beta S^\gamma = 0 \ .$$

TD SSC:

$U^a = u^a$

$$\left(\frac{dS^\alpha}{d\tau} + \varepsilon^\alpha{}_{\beta\gamma}\Omega^\beta S^\gamma\right)E_\alpha + \boldsymbol{\Upsilon} = 0 \ ,$$

$$\boldsymbol{\Upsilon} = \left[(S\cdot\mathbf{a})\,u^{\mathbf{A}} - (S\cdot\mathbf{F})\,\frac{p^{\mathbf{A}}}{M^2}\right]E_{\mathbf{A}} \ .$$

$u \cdot \boldsymbol{\Upsilon} = 0 \longrightarrow \boldsymbol{\Upsilon} = \Upsilon^\alpha E_\alpha$ $\qquad$ $S \cdot \boldsymbol{\Upsilon} = 0 \longrightarrow \omega \times S = \boldsymbol{\Upsilon}$

TD SSC:

$U^a = u^a$

$$\frac{dS^\alpha}{d\tau} + \varepsilon^\alpha{}_{\beta\gamma}\left(\Omega^\beta + \omega^\beta\right)S^\gamma = 0$$

Cartesian-like triads

SO frame:

$$(e_{\mathbf{1}}, e_{\mathbf{2}}, e_{\mathbf{3}}) = (e_{\mathbf{x}}, e_{\mathbf{y}}, e_{\mathbf{z}})\, R_{(e)}$$

$$R_{(e)} \equiv R\,(\theta, \phi)$$

D. Bini, A. Geralico, R.T. Jantzen, Phys. Rev. D **95**, 124022 (2017).

ZAMO frame:

$$(f_{\mathbf{1}}, f_{\mathbf{2}}, f_{\mathbf{3}}) = (f_{\mathbf{x}}, f_{\mathbf{y}}, f_{\mathbf{z}})\, R_{(f)}$$

$$R_{(f)} \equiv R\,(\theta, \phi)$$

$$R\,(\theta, \phi) = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

Boosted SO frame:

$$(E_{\mathbf{1}}, E_{\mathbf{2}}, E_{\mathbf{3}})\,(e, U)$$

$$= (E_{\mathbf{x}}, E_{\mathbf{y}}, E_{\mathbf{z}})\,(e, U)\, R_{(e)}$$

Boosted ZAMO frame:

$$(E_{\mathbf{1}}, E_{\mathbf{2}}, E_{\mathbf{3}})\,(f, U)$$

$$= (E_{\mathbf{x}}, E_{\mathbf{y}}, E_{\mathbf{z}})\,(f, U)\, R_{(f)}$$

Evolution equations for Cartesian-like triad components of the spin

FMP SSC: $\mathbb{S}^a = s^a$ TD SSC: $\mathbb{S}^a = S^a$ $(p_a \mathbb{S}^a = 0 = u_a \mathbb{S}^a)$

In the U-frame: $\mathbb{S} = \mathbb{S}^{\boldsymbol{\alpha}} E_{\boldsymbol{\alpha}} = \mathbb{S}^{\mathbf{i}} E_{\mathbf{i}}$, $\mathbb{S}^{\mathbf{0}} = 0.$

$(\alpha = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$, $\mathbf{i} = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\})$

$$\frac{d\mathbb{S}^{\mathbf{i}}}{d\tau} = -R^{\mathbf{i}}{}_{\alpha} \varepsilon^{\alpha}{}_{\beta\gamma} \Omega^{\beta}_{(prec)} \mathbb{S}^{\gamma}$$

$$\Omega^{\beta}_{(prec)} = \Omega^{\beta}_{(p)} + \epsilon \omega^{\beta} ,$$

$$\Omega^{\beta}_{(p)} = -\Omega^{\beta}_{(orb)} + \Omega^{\beta} , \quad (R^{-1})^{\alpha}{}_{\mathbf{j}} \frac{dR^{\mathbf{j}}{}_{\beta}}{d\tau} = \varepsilon^{\alpha}{}_{\gamma\beta} \Omega^{\gamma}_{(orb)} , \quad \Omega^{\alpha} = -\frac{1}{2} \varepsilon^{\alpha\beta\gamma} E_{\beta} \cdot \frac{DE_{\gamma}}{d\tau}$$

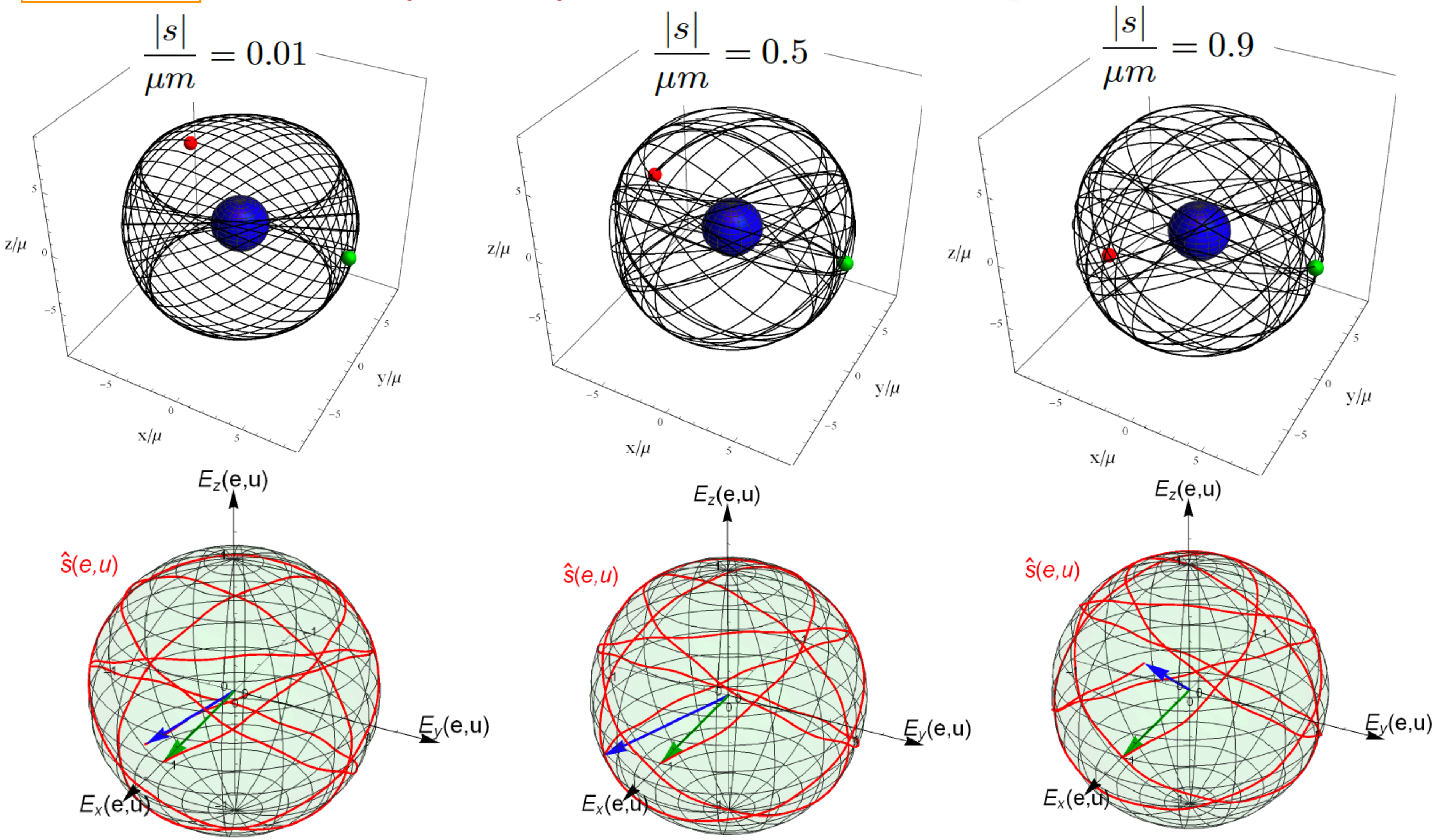
FMP SSC	TD SSC	
$U^a = u^a$	$U^a = p^a/M$	$U^a = u^a$
$\mathbb{S} = s$, $\epsilon = 0$	$\mathbb{S} = S$, $\epsilon = 0$	$\mathbb{S} = S$, $\epsilon = 1$

Spherical-like orbits (Kerr BH)

Coordinate space: $x = r\cos\phi\sin\theta\ ,\ y = r\sin\phi\sin\theta\ ,\ z = r\cos\theta\ .$

$a = 0.99\mu$

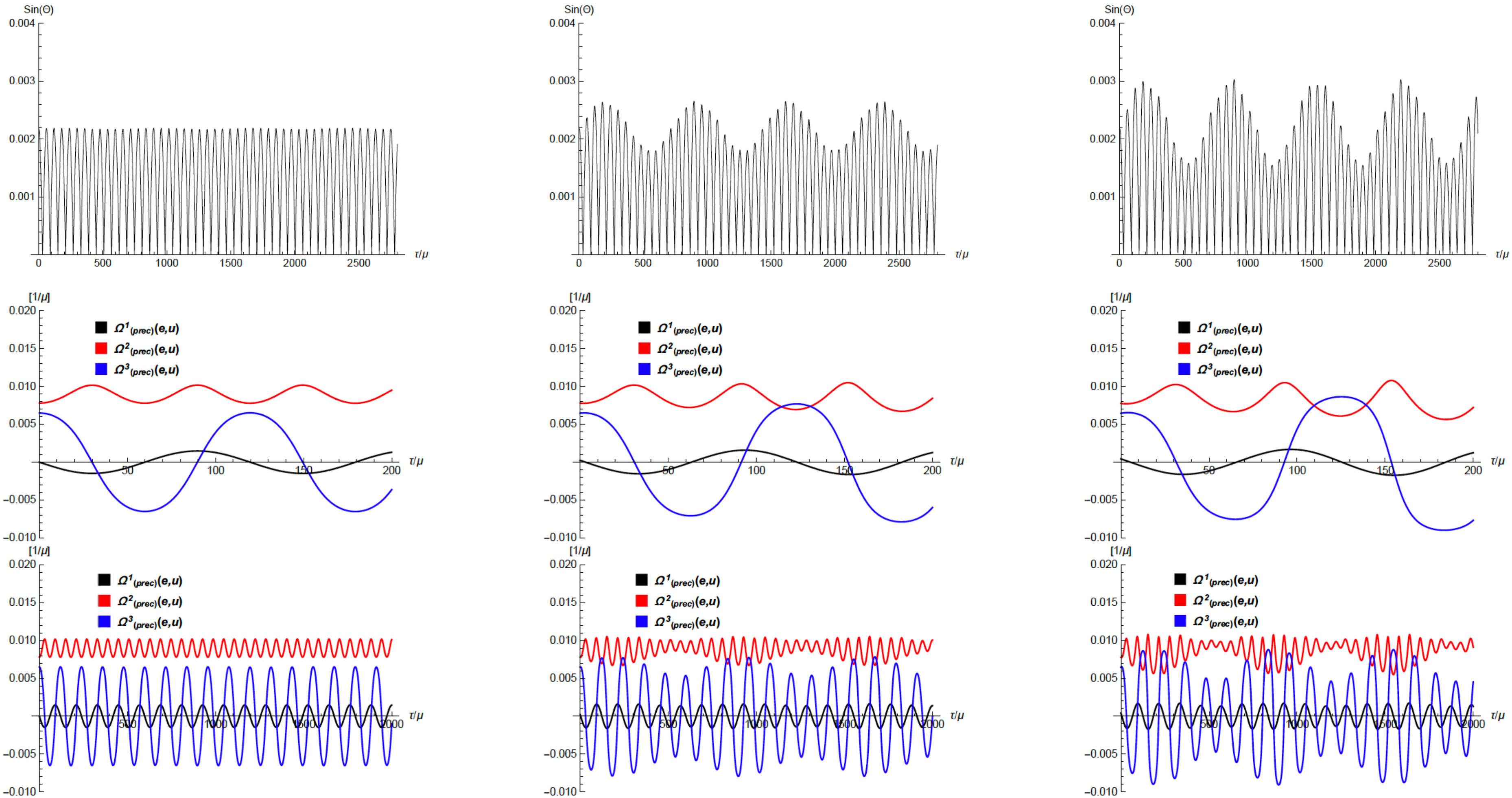
Increasing spin magnitude →



Spherical-like orbits (Kerr BH)

Increasing spin magnitude →

$\frac{|s|}{\mu m} = 0.01$ $\frac{|s|}{\mu m} = 0.5$ $\frac{|s|}{\mu m} = 0.9$

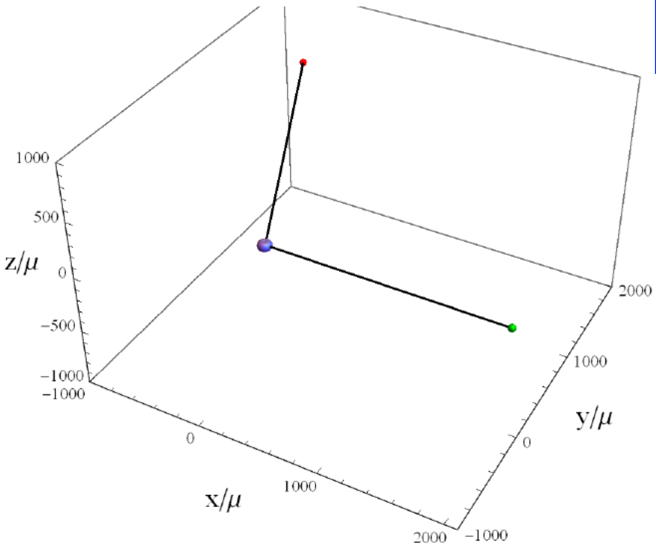


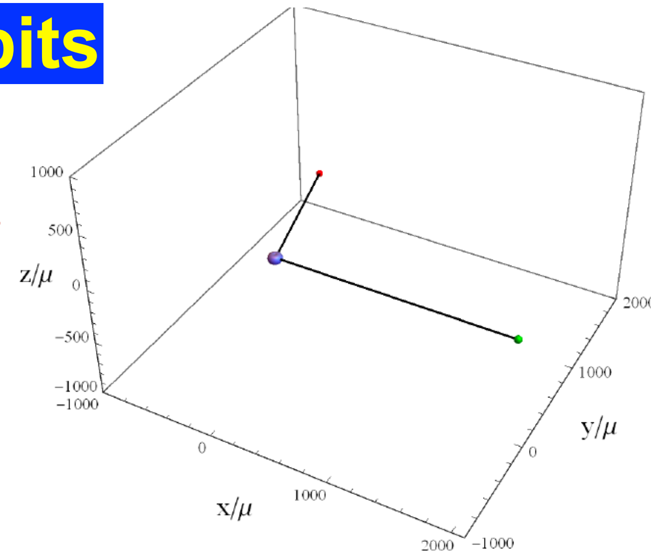
Unbound orbits

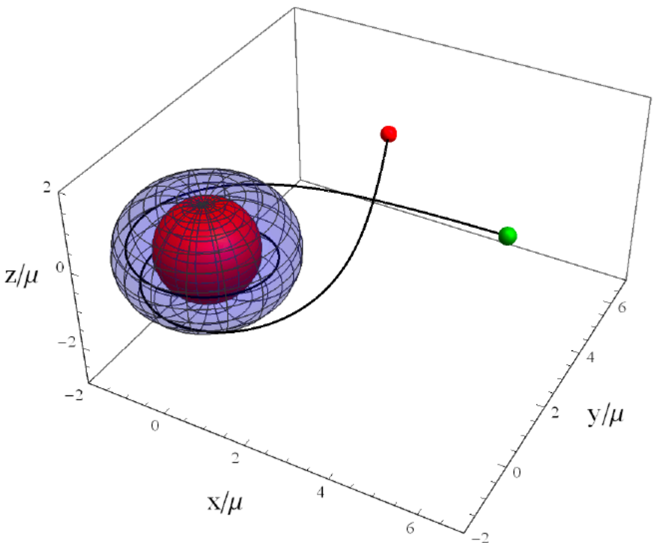
$a = 0.99\mu$

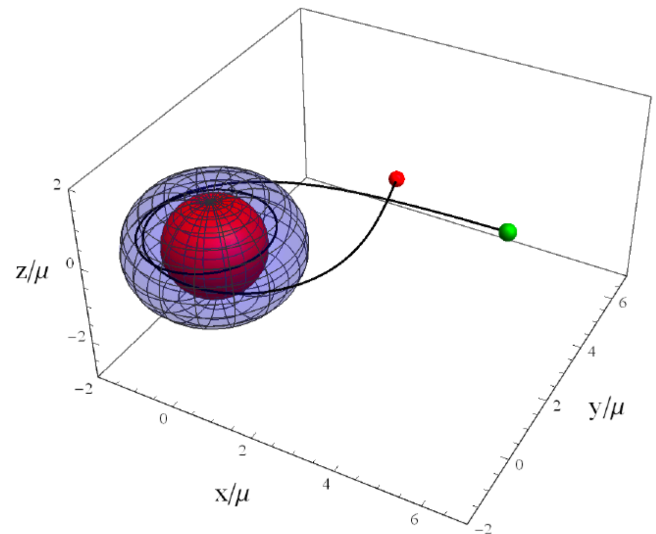
Initial spin directions are different.

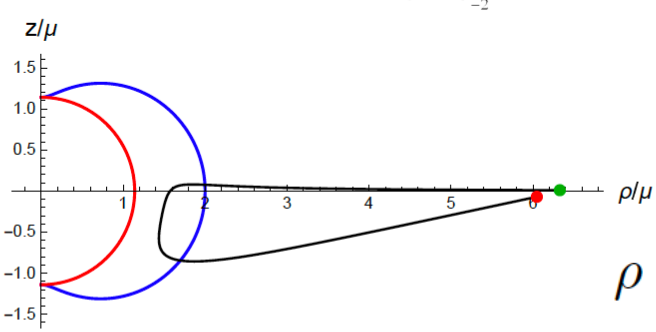
$$\frac{|S|}{\mu M} = 0.4$$



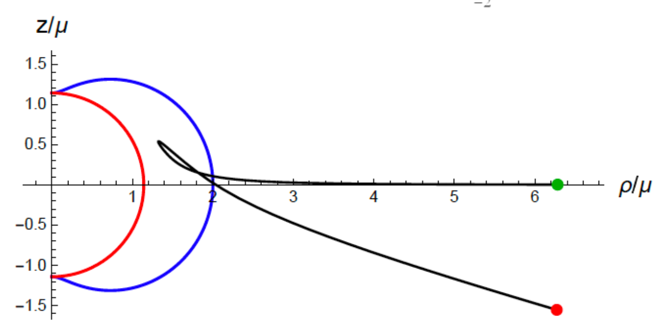


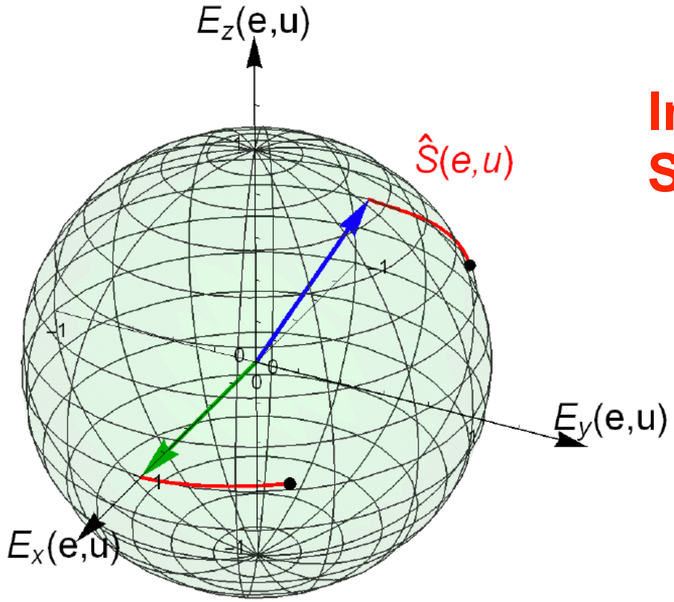




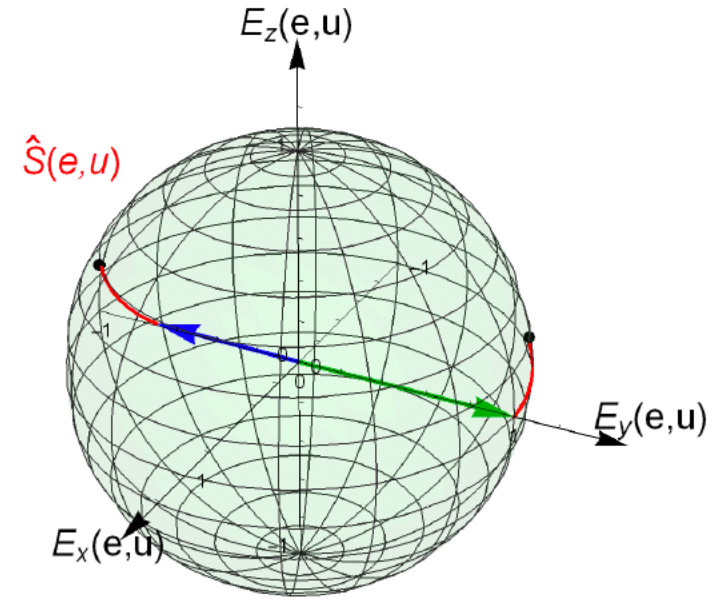


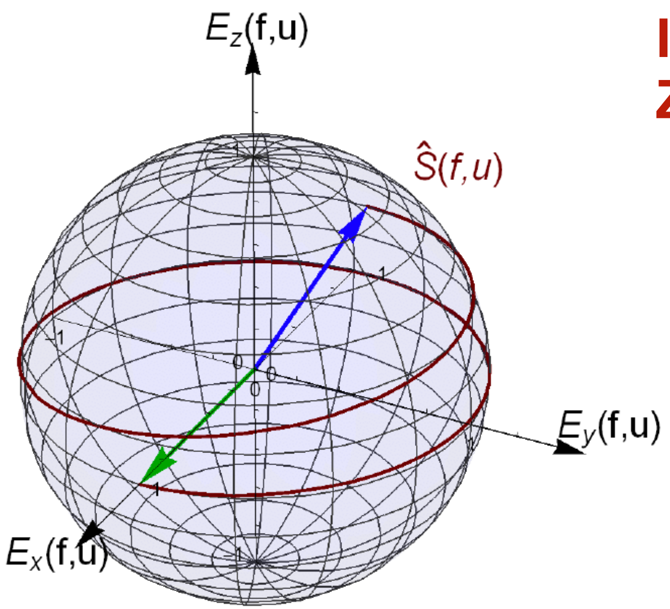
$$\rho = r \sin\theta$$



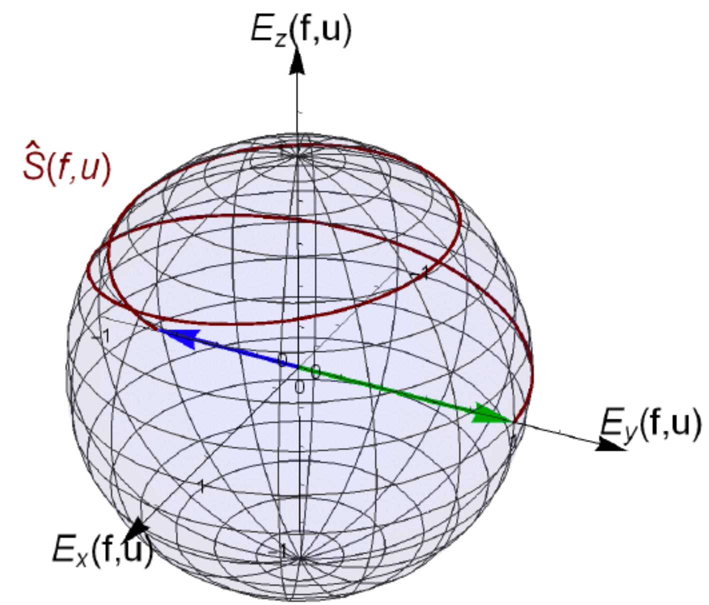


In boosted SO frame



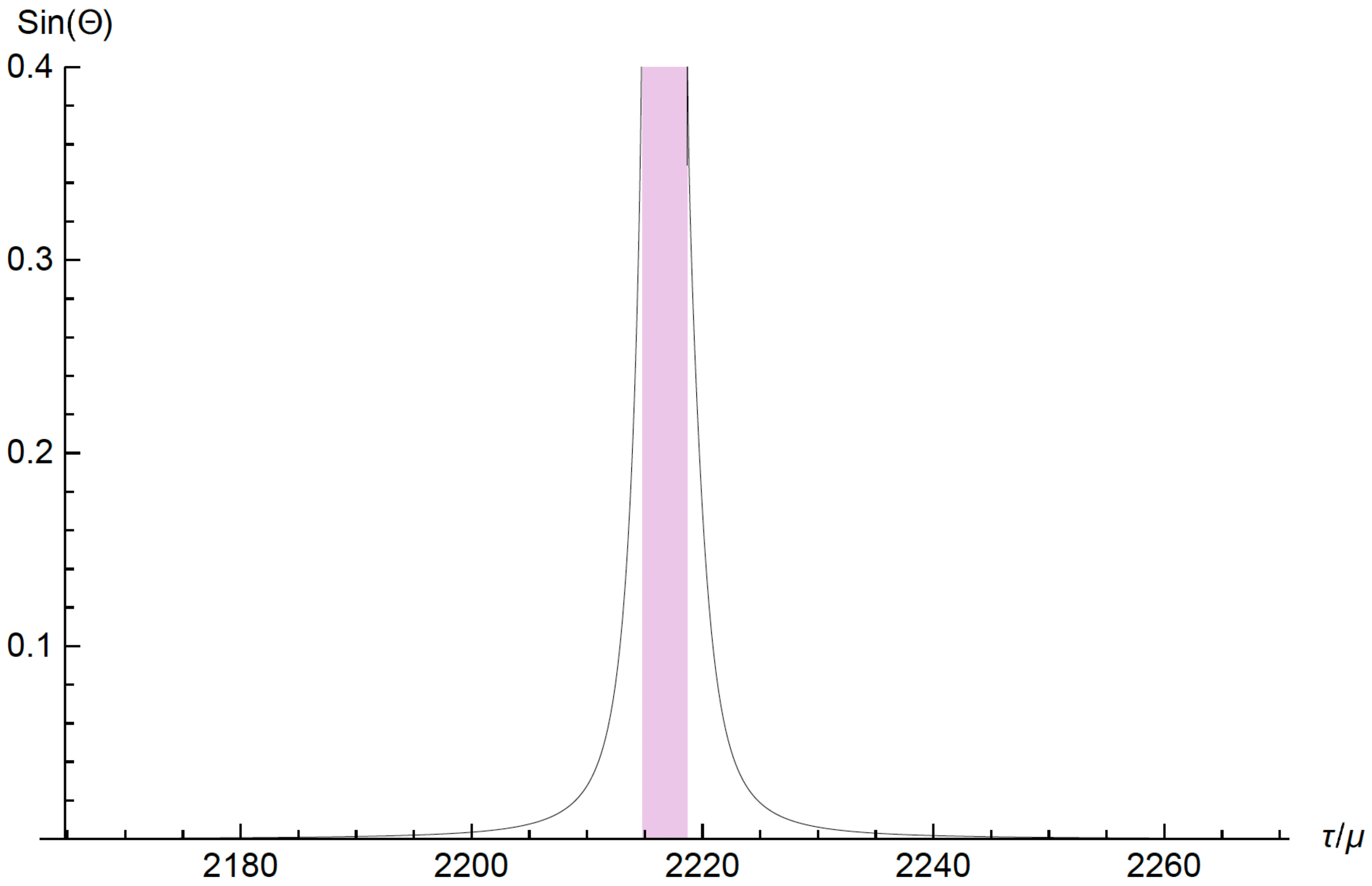


In boosted ZAMO frame



Rotation angle between the boosted SO and ZAMO frames

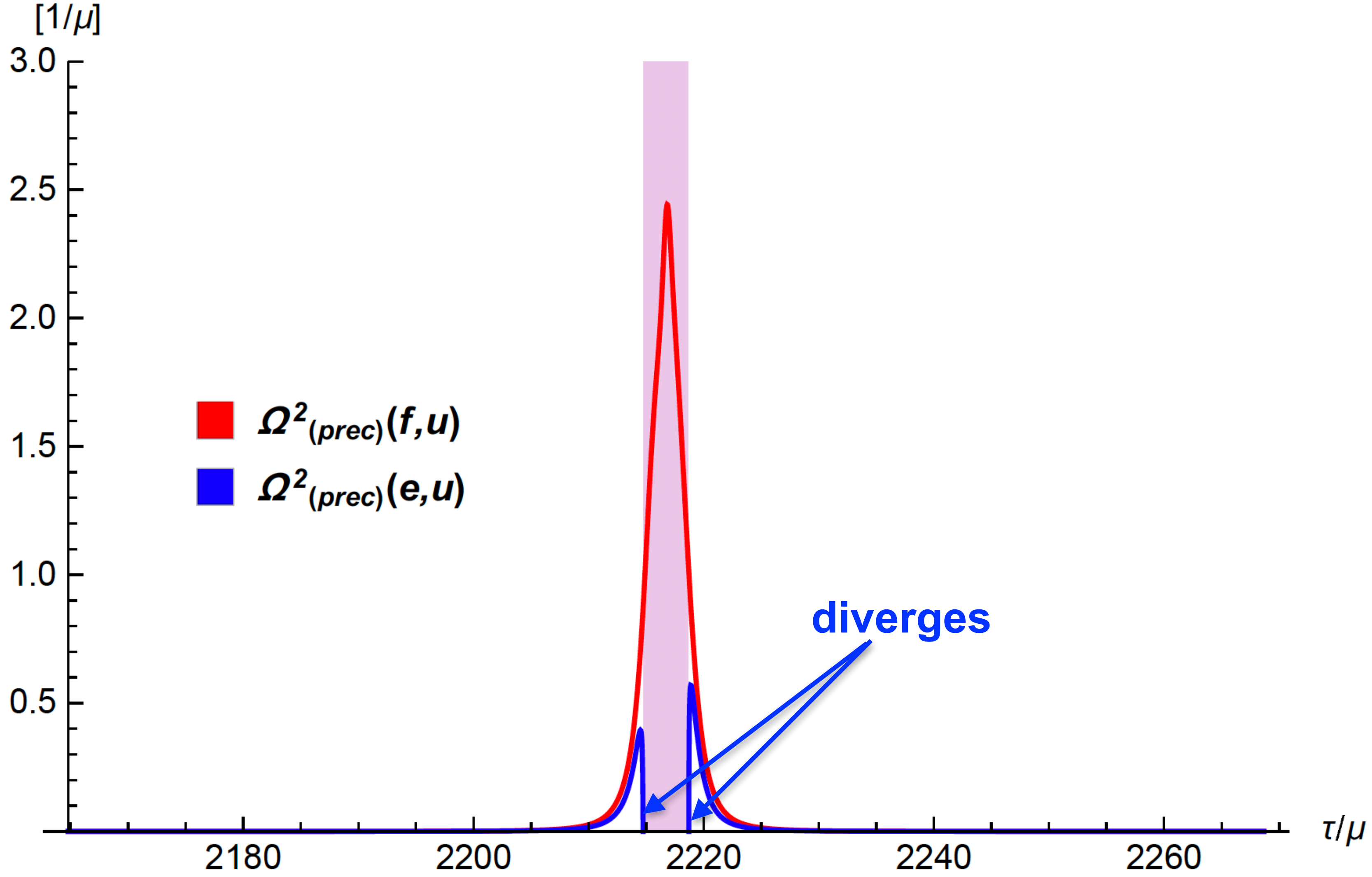
(In the left side hand case)

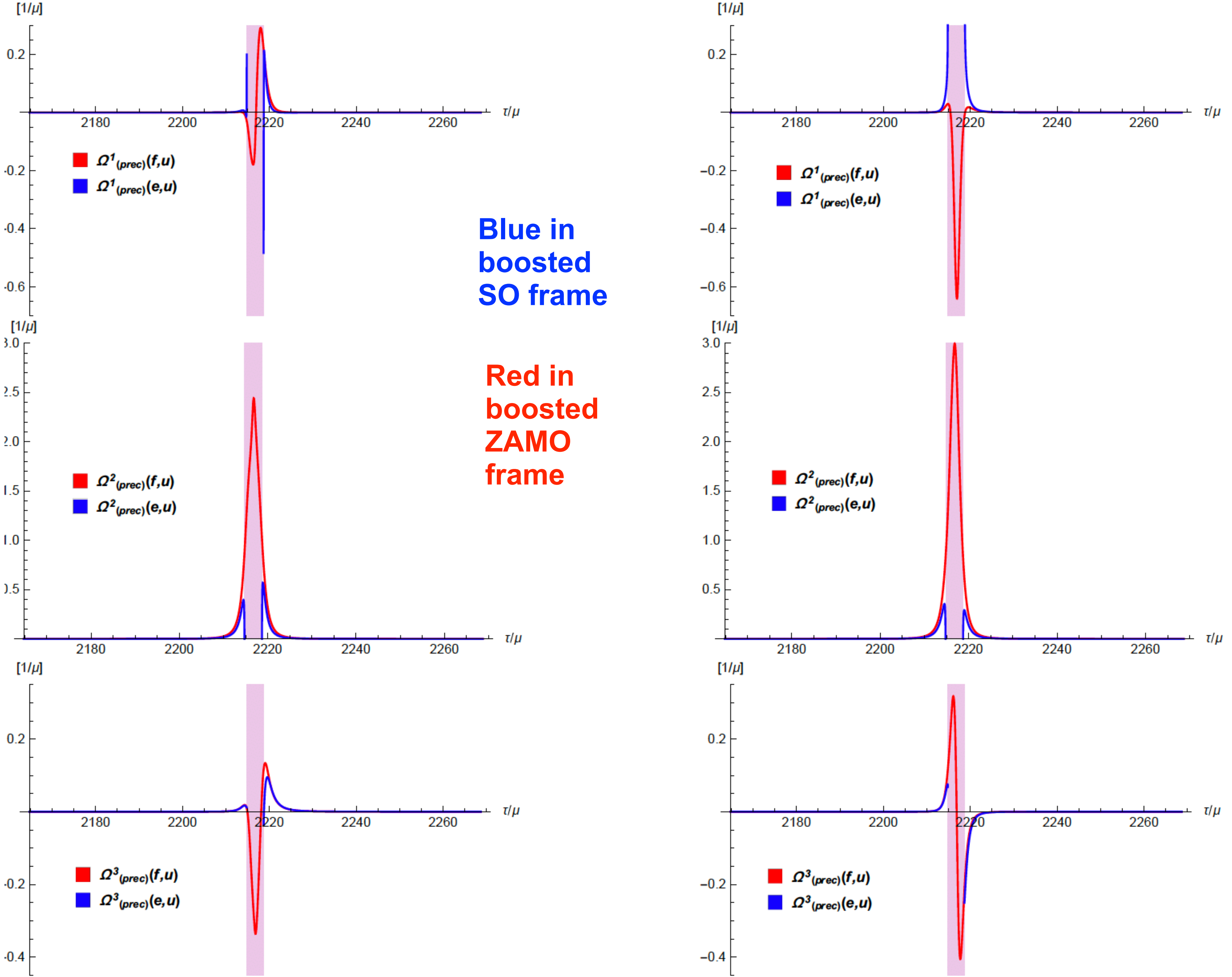


(In the left side hand case)

Precessional angular velocity

Blue in boosted SO frame **Red in boosted ZAMO frame**



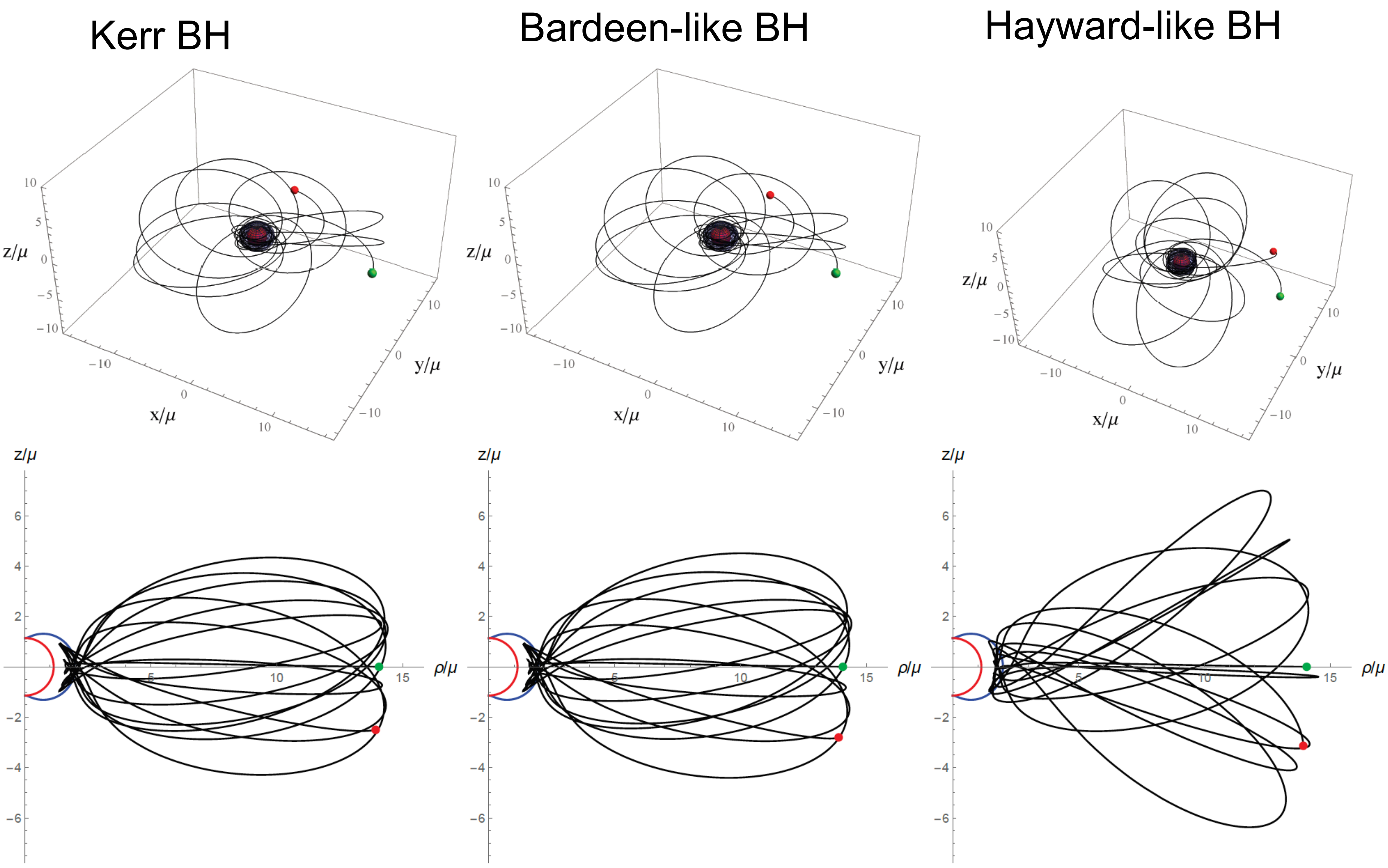


[1/μ]
τ/μ
2180
2200
2220
2240
2260
0.2
-0.2
-0.4
-0.6
$\Omega^1{}_{(prec)}(f,u)$
$\Omega^1{}_{(prec)}(e,u)$
Blue in boosted SO frame
3.0
2.5
2.0
1.5
1.0
0.5
$\Omega^2{}_{(prec)}(f,u)$
$\Omega^2{}_{(prec)}(e,u)$
Red in boosted ZAMO frame
$\Omega^3{}_{(prec)}(f,u)$
$\Omega^3{}_{(prec)}(e,u)$

Zoom-whirl orbits (Kerr and regular BHs)

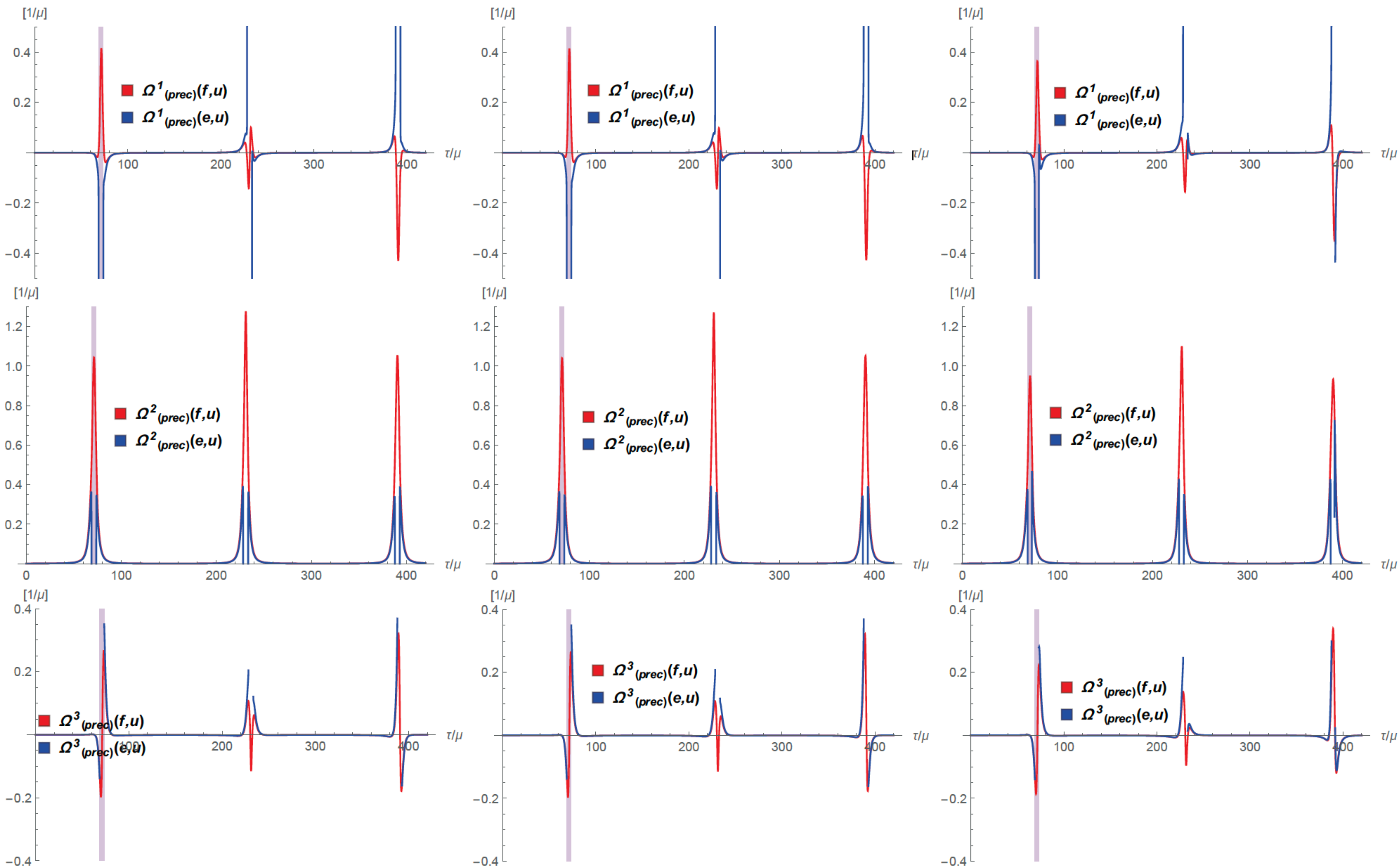
$a = 0.99\mu$

$q = 0.08$



Precessional angular velocity

Kerr BH | Bardeen-like BH | Hayward-like BH



Thank you for the attention

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