Obstacles to the quantization of general relativity using symplectic structures

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Recently, I proposed a novel toy-model for quantizing classical fields [1][2] similar in some ways to recent schemes like those presented in [3], [4], and [5], but built on a geometric foundation related more closely to the early polysymplectic formalism pioneered by Christian Gunther [6] and since refined and developed by many others. [7] contain the details for this formalism, but in a local coordinate system of the configuration space $\{x^\mu, \phi^I\}$ that leads to local (extended) phase space coordinates $\{x^\mu, \phi^I, \pi^\mu_I\}$ the main (poly)symplectic structures are

$$\omega = d\pi^\mu_I \wedge d\phi^I \otimes \frac{\partial}{\partial x^\mu} \tag{1}$$

$$\theta = \pi^\mu_I d\phi^I \otimes \frac{\partial}{\partial x^\mu} \tag{2}$$

$$\Pi = \frac{\partial}{\partial \phi^I} \wedge \frac{\partial}{\partial \pi^\mu_I} \otimes dx^\mu \tag{3}$$

$$v_\theta = \pi^\mu_I \frac{\partial}{\partial \pi^\mu_I} \tag{4}$$

Naively mimicking early attempts at geometric quantization (see, for example, [8]) in this field theory setting gives the quantization map

$$Q_v(f) = f - v_\theta(f) + \mathrm{i}\hbar\, \iota_v \Pi(df) \tag{5}$$

This quantization map is – it must be said again – only a toy model for field theory quantization. It does, however, give a few interesting results. For example, the choice of $v = \frac{\partial}{\partial t}$ gives an integrated commutation relation for a scalar field that reads

$$[Q(\phi), Q(\pi^0)] = \mathrm{i}\hbar = \int \star g(v)[\hat{\phi}(x), \hat{\pi}^0(y)] \tag{6}$$

where $\hat{\phi}(x)$ and $\hat{\pi}^0(y)$ are the usual scalar field and momentum operators of the standard canonical approach to quantum field theory.

However, the primary merit of this toy model is that it is a simple, well-defined tensorial map that captures a few of the important features of field theory quantization. Because the process is both simple and inherently geometric, there is – perhaps quite surprisingly – no technical obstacle to applying it to the case of general relativity. Taking the configuration space to be the symmetric sections of $T^*M \otimes T^*M$, we find that (on a suitably reduced state space):

$$Q(g_{\mu\nu}) = g_{\mu\nu} \tag{7}$$

$$Q(\pi^{0\mu\nu}) = -\mathrm{i}\hbar \frac{\partial}{\partial g_{\mu\nu}} \tag{8}$$

However, there is a serious issue unrelated to the quantization process that makes it more-or-less impossible to take this result seriously: the Legendre transformation from the Einstein-Hilbert action $\mathscr{L}_{EH}$ to $\mathscr{H}$ that allows us to define the classical theory in the formalism of [7] is not invertible, lacking information about the second derivatives of the metric that appear in the action. We can try to remedy this problem by changing to the Palatini or ADM perspectives, but the problem is not solved. We therefore find that the primary obstacle to the quantization of general relativity in this formalism is purely classical in nature! Some issues associated with applying the alternative approaches outlined in [9] are briefly discussed.

Bibliography

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