Obstacles to the quantization of general relativity using symplectic structures

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Overview

+ The problem
+ Classical field theory with symplectic structures
+ Quantization with symplectic structures
+ Obstacles for general relativity

Statement of the problem

- General relativity is not perturbatively renormalizable
- Normal quantum field theory methods fail
- Other quantum field theory methods might succeed

Wish list for polysymplectic Hamiltonian field theory for quantum field theory

+ Right equations of motion for real physical systems
+ Fully differential geometric
+ **Use only polysymplectic structures with direct analogs in Hamiltonian particle theory**

Configuration, (extended) phase, and "symplectic" spaces

$$\epsilon : E \to M \mid e = x^\alpha e_\alpha + \phi^I e_I$$

$$V_e E := \{u \in T_e E \mid T\epsilon(u) = 0\}$$

$$P = V^* E \underset{E}{\otimes} TM \mid p = x^\alpha e_\alpha + \phi^I e_I + \pi_I^\alpha \, d\phi^I \otimes \frac{\partial}{\partial x^\alpha}$$

$$S = VP \underset{P}{\otimes} T^* M \mid s = x^\alpha e_\alpha + \phi^I e_I + \pi_I^\alpha \, d\phi^I \otimes \frac{\partial}{\partial x^\alpha} + v_\alpha^I \frac{\partial}{\partial \phi^I} \otimes dx^\alpha + s_{I\beta}^\alpha \frac{\partial}{\partial \pi_I^\alpha} \otimes dx^\beta$$

Polysymplectic structures

$$\theta_p(s) := p \circ T\pi(s) \mid \theta = \pi_I^\alpha d\phi^I \otimes \frac{\partial}{\partial x^\alpha}$$

$$\omega(u, v, \beta) := d(\theta \lrcorner \beta)(u, v) \mid \omega = d\pi_I^\alpha \wedge d\phi^I \otimes \frac{\partial}{\partial x^\alpha}$$

$$\mathscr{S}_f := \{s \mid \omega(s, v) = df(v)\ \forall\ v\}$$

$$\Pi := \Pi(df) \in \mathscr{S}_f \mid \Pi = \frac{\partial}{\partial \phi^I} \wedge \frac{\partial}{\partial \pi_I^\alpha} \otimes dx^\alpha$$

$$V := \Pi(\theta) \mid V = \pi_I^\alpha \frac{\partial}{\partial \pi_I^\alpha}$$

$s \in \Gamma(P, S)$, $u, v \in \Gamma(P, VP)$, Π anti-symmetric in its first two arguments

Hamilton's equations

$$V\gamma : TP \to VP \mid u \mapsto u - T(\gamma \circ \epsilon \circ \pi) \mid V\gamma = (d\pi_I^\alpha - \frac{\partial \gamma_I^\alpha}{\partial x^\beta} dx^\beta) \otimes \frac{\partial}{\partial \pi_I^\alpha} + (d\phi^I - \frac{\partial \gamma^I}{\partial x^\beta} dx^\beta) \otimes \frac{\partial}{\partial \phi^I}$$

$$\omega(V\gamma, v) = dH(v) \mid \boxed{\frac{\partial \gamma^I}{\partial x^\alpha} = \frac{\partial H}{\partial \pi_I^\alpha} \; , \; \frac{\partial \gamma_I^\alpha}{\partial x^\alpha} = -\frac{\partial H}{\partial \phi^I}}$$

A simple quantization map

$$Q(f) = f - \Sigma(\theta) f + i\hbar \Sigma(df, -, \frac{\partial}{\partial t})$$

$$Q(q^a) = q^a - 0 + \boxed{i\hbar \frac{\partial}{\partial p_a}}$$

$$Q(p_a) = p_a - p_a - i\hbar \frac{\partial}{\partial q^a}$$

$$Q(\phi) = \phi - 0 + \boxed{i\hbar \frac{\partial}{\partial \pi^0}}$$

$$Q(\pi^0) = \pi^0 - \pi^0 - i\hbar \frac{\partial}{\partial \phi}$$

Space of states?

$$\Psi \in C^\infty(P, \mathbb{C}) \mid d\Psi(v) = 0 \;\forall\; v \mid d\Psi = \frac{\partial \Psi}{\partial x^\alpha} dx^\alpha + \frac{\partial \Psi}{\partial \phi^I} d\phi^I + 0$$

$$\Psi = \Psi(x^\alpha, \phi^I)$$

A complicated quantization map

$$Q(f) := f - Vf + \frac{1}{2}\Big(V^2 f - Vf\Big) + i\hbar \lim_{\epsilon\to 0} \frac{2Vf - V^2 f}{2Vf - V^2 f + \epsilon} \Pi(df, -, \frac{\partial}{\partial t}) - \hbar^2 \lim_{\epsilon \to 0} \frac{V^2 f - Vf}{V^2 f - Vf + \epsilon} \Delta$$

$$Q(H_{SHO}) = \alpha g_{ab} q^a q^b \boxed{- \hbar^2 g^{ab} \frac{\partial}{\partial q^a} \frac{\partial}{\partial q^b}}$$

$$Q(H_{KG}) = \beta \phi^2 \boxed{- \hbar^2 \frac{\partial^2}{\partial \phi^2}}$$

Issues with quantization of fields

- **Integrated** commutation relation!
- Right operators?
- Right states?
- Where does the vector field in our quantization map come from?

Quantizing general relativity

$$E = T^*M \otimes T^*M$$

$$\Pi = \frac{\partial}{\partial g_{\alpha\beta}} \wedge \frac{\partial}{\partial \pi^{\alpha\beta\gamma}} \otimes dx^\gamma$$

$$V = \pi^{\alpha\beta\gamma} \frac{\partial}{\partial \pi^{\alpha\beta\gamma}}$$

$$\Psi = \Psi(x^\alpha, g_{\alpha\beta})$$

$$Q(g_{\alpha\beta}) = g_{\alpha\beta}$$

$$Q(\pi^{0\alpha\beta}) = -\mathrm{i}\hbar \frac{\partial}{\partial g_{\alpha\beta}}$$

Issues with quantizing general relativity

- What vector field should we use to define **Q**?
- Hamiltonian not well-defined (Legendre transformation)
- Cannot take the quantization process seriously if the classical theory isn't well defined!
- Purely **classical** problems!

Solutions?

- Different starting geometries?
- Extended Legendre transformations?
- Different Lagrangians?
- Eliminate the Lagrangian and Legendre transform?
- Other approaches?

Questions?

- For closely related work, please see…

  - Günther, Christian. "The polysymplectic Hamiltonian formalism in field theory and calculus of variations. I. The local case." *Journal of differential geometry* 25.1 (1987): 23-53.
  - Struckmeier, Jürgen, and Andreas Redelbach. "Covariant Hamiltonian field theory." *International Journal of Modern Physics E* 17.03 (2008): 435-491.
  - Kanatchikov, Igor V. "Toward the Born-Weyl quantization of fields." *International journal of theoretical physics* 37.1 (1998): 333-342.
  - Magnano, Guido, Marco Ferraris, and Mauro Francaviglia. "Legendre transformation and dynamical structure of higher-derivative gravity." *Classical and Quantum Gravity* 7.4 (1990): 557.Different Lagrangians?

Appendix

More words on symplectic structures

The tautological tensor

Intrinsic definition

$$\theta_p(u) := p \circ T\pi(u)$$

$$\theta : S \to \mathbb{R} \mid (x^\alpha, \phi^I, \pi_I^\alpha, v_\alpha^I, \sigma_{I\beta}^\alpha) \mapsto p_I^a v_\alpha^I$$

In local canonical coordinates:

$$\theta_p = \pi_I^\alpha(p)\, d\phi^I \otimes \frac{\partial}{\partial x^\alpha}$$

The polysymplectic structure (Part I)

Intrinsic definition (first try):

$$\omega(u, v, \beta) := d(\theta \lrcorner \beta)(u, v)$$

In canonical coordinates:

$$d(\theta \lrcorner \beta) = d(\beta_\alpha \pi_I^\alpha d\phi^I) = \beta_\alpha d\pi_I^\alpha \wedge d\phi^I + \frac{\partial \beta_\alpha}{\partial x^\beta} \pi_I^\alpha dx^\beta \wedge d\phi^I$$

(depends on β!)

The polysymplectic structure (Part II)

- Solution: restrict to vertical vector fields:

$$d(\theta \lrcorner \beta)(u, v) = \beta_\alpha (u^I v_I^\alpha - u_I^\alpha v^I)$$

Now

$$\omega = d\pi_I^\alpha \wedge d\phi^I \otimes \frac{\partial}{\partial x^\alpha}$$

Hamilton's field equations (Part I)

Vertical differential of a section:

$$V\gamma : TP \to VP \mid u \mapsto u - T(\gamma \circ \epsilon \circ \pi)$$

In coordinates:

$$V\gamma = (d\pi_I^\alpha - \frac{\partial \gamma_I^\alpha}{\partial x^\beta} dx^\beta) \otimes \frac{\partial}{\partial \pi_I^\alpha} + (d\phi^I - \frac{\partial \gamma^I}{\partial x^\beta} dx^\beta) \otimes \frac{\partial}{\partial \phi^I}$$

Hamilton's field equations (Part II)

Solution sections must satisfy:

$$\omega(V\gamma, v) = dH(v)$$

for all vertical vector fields **u**

Gives Hamilton's equations

$$\frac{\partial \phi^I}{\partial x^\alpha} = \frac{\partial H}{\partial \pi_I^\alpha} \qquad \frac{\partial \pi_I^\alpha}{\partial x^\alpha} = -\frac{\partial H}{\partial \phi^I}$$

Poisson brackets (Part I)

For each function f on **P**, there exists a family of sections $\mathbf{S}_f$ such that:

$$\omega(s_f, v) = df(v) = v_I^\alpha s_\alpha^I - v^I s_{I\alpha}^\alpha$$

In canonical coordinates:

$$s_f = -\frac{\partial f}{\partial \pi_I^\alpha}\frac{\partial}{\partial \phi^I} \otimes dx^\alpha + \frac{\partial f}{\partial \phi^I}\frac{\partial}{\partial \pi_I^\alpha} \otimes dx^\alpha + \sigma_{TF}{}_{I\beta}^{\alpha}\frac{\partial}{\partial \pi_I^\alpha} \otimes dx^\beta$$

(the last components must be trace-free)

Poisson brackets (Part II)

Define a new tensor via:

$$\Pi(df) \in \mathscr{S}_f$$

for all functions on the phase space

Imposing anti-symmetry gives:

$$\Pi = \frac{\partial}{\partial \phi^I} \wedge \frac{\partial}{\partial \pi_I^\alpha} \otimes dx^\alpha$$

(no contribution from the trace-free components!)

Poisson brackets (Part III)

Define the Poisson bracket via:

$$\{f,\, g\} := \Pi(df, dg)$$

for all functions on the phase space

In canonical coordinates:

$$\{f,\, g\} = \left(\frac{\partial f}{\partial \phi^I}\frac{\partial g}{\partial \pi_I^\alpha} - \frac{\partial f}{\partial \pi_I^\alpha}\frac{\partial g}{\partial \phi^I}\right) dx^\alpha$$