Relativistic QM as a Theory of Wave Functions on Configuration Spacetime

Matthias Lienert*

February 14, 2019

The most fundamental object in quantum mechanics is the quantum state $|\psi\rangle$. While it is usually regarded as an element of an abstract, timeless Hilbert space, one needs a particle position representation to make contact with space and time. For one-body equations such as the Dirac equation, such a representation is given by a wave function

$$\psi : \mathbb{M} \to \mathbb{C}^4, \quad x \mapsto \psi(x), \tag{1}$$

where $\mathbb{M}$ is Minkowski spacetime.

However, for many (N) particles, if a particle-position representation is at all discussed, one normally uses the wave function in the Schrödinger picture which is a map

$$\varphi : \mathbb{R} \times \mathbb{R}^{3N} \to \mathbb{C}^k, \quad (t, \mathbf{x}_1, ..., \mathbf{x}_N) \mapsto \varphi(t, \mathbf{x}_1, ..., \mathbf{x}_N). \tag{2}$$

Here, the first factor $\mathbb{R}$ stands for time and the second, $\mathbb{R}^{3N}$, for the (timeless) configuration space of N particles in $\mathbb{R}^3$. Embarrassingly, *φ is not a Lorentz covariant object.*

In this talk, we discuss the idea (first suggested by Dirac in 1932 but investigated in detail only in recently) that in a relativistic context, the wave function should instead be map on *configuration spacetime*, i.e.

$$\psi : \Omega \subset \mathbb{M}^N \to \mathbb{C}^k, \quad (x_1, ..., x_N) \mapsto \psi(x_1, ..., x_N). \tag{3}$$

Here, Ω is a suitable Poincare-invariant subset of $\mathbb{M}^N$, most naturally the set of space-like configurations. Because of the occurrence of many x_i^0-coordinates, ψ is called a *multi-time wave function*. The relation to φ is straightforwardly given by

$$\varphi(t, \mathbf{x}_1, ..., \mathbf{x}_N) = \psi(t, \mathbf{x}_1, ..., t, \mathbf{x}_N). \tag{4}$$

We thus obtain a natural, manifestly covariant generalization of the single-time wave function (2). However, several important questions arise:

1. How does ψ evolve in the many time coordinates?
2. Can one re-formulated quantum field theory using multi-time wave functions?
3. What is the physical meaning of ψ? Does $|\psi|^2$ represent a probability density?
4. Does a multi-time wave function provide new resources to formulate interacting relativistic quantum theories?

*Fachbereich Mathematik, Eberhard-Karls-Universität, Auf der Morgenstelle 10, 72076 Tübingen, Germany. E-mail: matthias.lienert@uni-tuebingen.de

This talk shall give a glimpse into each of these topics, taking into account progress achieved in recent years. Regarding 1, the idea is that ψ needs to satisfy a system of N PDEs, one for each time coordinate:

$$\begin{aligned} i\partial_{x_1^0}\psi &= H_1\psi \\ &\vdots \\ i\partial_{x_N^0}\psi &= H_N\psi, \end{aligned} \tag{5}$$

where the H_N are differential operators (related to the Hamiltonian). The main difficulty is that the evolutions in the many time directions have to commute. One needs to satisfy a restrictive consistency condition which makes interaction difficult to achieve. For example, covariant potentials are excluded. Recently, however, there has been progress in rigorously constructing interacting models, e.g., using contact interactions.

Concerning 2, one can extend the multi-time wave to quantum field theory using a Fock space construction. It becomes a sequence $\psi = (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, ...)$ of N-particle wave functions. Each $\psi^{(n)}$ then satisfies a system of multi-time equations (5) where now the right hand side may involve interaction terms which couple $\psi^{(n)}$ with $\psi^{(m)}$, $m \neq n$. This coupling corresponds to particle creation and annihilation (which can be regarded a type of contact interaction).

With regard to 3., it is a natural postulate analogous to the usual Born rule in one frame that for a spacelike hypersurface Σ with unit normal vector field n, the probability to detect N particles, each in an infinitesimal 3-volume $d\sigma(x_i)$ around $x_i \in \Sigma$ is given by:

$$\text{Prob}(X_1 \in d\sigma(x_1), ..., X_N \in d\sigma(x_N)) = |\psi|^2_\Sigma(x_1, ..., x_N)\, d\sigma(x_1) \cdots d\sigma(x_N), \tag{6}$$

where $|\psi|^2_\Sigma$ is a quadratic expression in ψ, taking into account the curvature of Σ. For example, for N Dirac particles, one has: $|\psi|^2_\Sigma(x_1, ..., x_N) = \overline{\psi}(x_1, ..., x_N)\gamma^{\mu_1} \otimes \cdots \otimes \gamma^{\mu_N}\psi(x_1, ..., x_N)\, n_{\mu_1}(x_1) \cdots n_{\mu_N}(x_N)$. The crucial point, however, is that it may not be conceptually adequate to postulate this *curved Born rule*. The Born rule in one frame should in principle be sufficient to describe the position statistics of all conceivable arrangements of particles, hence of all conceivable measurement devices, in particular detectors placed along curved hypersurfaces Σ. The statement (6) is thus either false or can be derived as a theorem. By approximating Σ with pieces of equal-time hyperplanes (where the usual Born rule and measurement postulates are assumed to hold), it has recently been shown that the latter is the case. We shall give a glimpse into the argument.

Finally, concerning 4., a multi-time wave function exists on space-time configurations which are not simultaneous in any frame. It thus contains more information than all the equal-time wave functions of all Lorentz frames taken together. This additional freedom can be used to formulate novel kinds of interacting quantum dynamics which cannot so easily be expressed in the other pictures of quantum theory. For example, *direct interactions along light cones*, such as in the Wheeler-Feynman formulation of classical electrodynamics, can be expressed using ψ on light-like configurations (x_1, x_2), $|x_1^0 - x_2^0| = |\mathbf{x}_1 - \mathbf{x}_2|^2$. This naturally leads to integral equations for a multi-time wave functions, related (but not identical to) the Bethe-Salpeter equation of quantum field theory. Time permitting, we shall present the main new ideas about direct interactions in quantum theory here.