The evolutions of spinning bodies moving in rotating black hole spacetimes

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The evolution of the extended spinning bodies are governed by the Mathisson-Papapetrou-Dixon (MPD) equations. The MPD system becomes closed with a spin supplementary condition (SSC). In the presentation the spin dynamics is investigated both with Frenkel-Mathisson-Pirani (FMP) and Tulczyjew-Dixon (TD) spin supplementary conditions. The spinning body moves in a rotating black hole (BH) spacetime which is given by the Kerr geometry or by a regular BH geometry (e.g. the rotating Bardeen-like and Hayward-like spacetimes). The model describes black hole binary systems with small mass ratio which can be the source for gravitational waves in the frequency sensitivity range of the planned LISA - Laser Interferometer Space Antenna.

The Mathisson-Papapetrou-Dixon (MPD) equations [1–4] read as

$$\frac{Dp^a}{d\tau} \equiv u^c \nabla_c p^a = -\frac{1}{2} R^a{}_{bcd} u^b S^{cd} \ , \ \frac{DS^{ab}}{d\tau} \equiv u^c \nabla_c S^{ab} = p^a u^b - u^a p^b \ .$$

Here ∇_c is the covariant derivative, $R^a{}_{bcd}$ is the Riemann tensor of the background spacetime, while p^a is the four-momentum of the moving body, S^{ab} is its spin tensor and u^a (with $u^a u_a = -1$) is the four-velocity of the representative point for the extended body which is determined by the SSC. In this abstract the FMP SSC [1, 5, 6] is introduced: $u_a S^{ab} = 0$. The spin vector is defined by $s^a = -\frac{1}{2}\eta^{abcd} u_b S_{cd}$, with 4-dimensional Levi-Civita tensor η_{abcd} which is totally antisymmetric and $\eta_{0123} = \sqrt{-g}$, where g is the determinant of the background metric. The MPD equations with FMP SSC give two constants of motion: $s^2 = s_a s^a = \frac{1}{2} S_{cd} S^{cd}$ and $m = -u_a p^a$.

We describe the spin dynamics in the comoving frame obtained by boost transformations from the frames of the static observers (SO). In this abstract we consider a spinning body moving in the Kerr spacetime. Then the SO's frame is given by

$$e_{\mathbf{0}} = u_{(SO)} = \frac{1}{\sqrt{-g_{tt}}}\partial_t \ , \ e_{\mathbf{1}} = \sqrt{\frac{\Delta}{\Sigma}}\partial_r \ , \ e_{\mathbf{2}} = \frac{\partial_\theta}{\sqrt{\Sigma}} \ , \ e_{\mathbf{3}} = -\frac{1}{\sqrt{\Delta}}\left(\frac{a\mathcal{B}\sin\theta}{\Sigma\sqrt{-g_{tt}}}\partial_t - \frac{\sqrt{-g_{tt}}}{\sin\theta}\partial_\phi\right) \ ,$$

with metric functions

$$\Sigma = r^2 + a^2\cos^2\theta \ , \ \Delta = r^2 + a^2 - 2\mu r \ , \ \mathcal{B} = r^2 + a^2 - \Delta \ , \ \mathcal{A} = \left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2\theta \ ,$$

where t, r, θ, ϕ are the Boyer-Lindquist coordinates, and a is the rotation parameter and μ is the mass parameter of the central black hole. The comoving frame [7] is

$$E_{\mathbf{0}}(e,u) \equiv u = \Gamma_{(S)}\left(e_{\mathbf{0}} + \mathbf{v}_{(S)}\right) \ , \ E_\alpha(e,u) = e_\alpha + \frac{u \cdot e_\alpha}{1 + \Gamma_{(S)}}\left(u + u_{(SO)}\right) \ ,$$

with frame indices $\alpha = \{\mathbf{1},\mathbf{2},\mathbf{3}\}$, relative spatial velocity $\mathbf{v}_{(S)} = \Gamma_{(S)}^{-1} u - u_{(SO)}$ and Lorentz factor $\Gamma_{(S)} = -u \cdot u_{(SO)}$. The dot denotes the inner product with respect to the background spacetime metric. Following Ref. [7], we introduce Cartesian-like 3-bases $e_{\mathbf{x}}$, $e_{\mathbf{y}}$ and $e_{\mathbf{z}}$ in the local rest spaces of the static observers as $(e_{\mathbf{1}}, e_{\mathbf{2}}, e_{\mathbf{3}}) = (e_{\mathbf{x}}, e_{\mathbf{y}}, e_{\mathbf{z}})\, R(\theta,\phi)$, where $R(\theta,\phi)$ is the same rotation matrix which relates the Cartesian and spherical coordinates in the 3-dimensional Euclidean space. The orbit of the spinning body is represented by the corresponding Cartesian-like coordinates $x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$ and $z = r\cos\theta$. From the MPD equations we find the following the evolution equations for the Cartesian-like frame components of the spin vector:

$$\frac{ds^{\mathbf{i}}}{d\tau} = -R^{\mathbf{i}}{}_\alpha \varepsilon^\alpha{}_{\beta\gamma} \Omega^\beta_{(prec)} s^\gamma \ ,$$

where $\mathbf{i} = \{\mathbf{x},\mathbf{y},\mathbf{z}\}$ and α, β, $\gamma = \{\mathbf{1},\mathbf{2},\mathbf{3}\}$. In addition $\Omega^\beta_{(prec)} = -\Omega^\beta_{(orb)} + \Omega^\beta$ with $\Omega^{\mathbf{1}}_{(orb)} = \cos\theta\dot{\phi}$, $\Omega^{\mathbf{2}}_{(orb)} = -\sin\theta\dot{\phi}$, $\Omega^{\mathbf{3}}_{(orb)} = \dot{\theta}$ and $\Omega^{\mathbf{1}} = E_{\mathbf{3}} \cdot DE_{\mathbf{2}}/d\tau$, $\Omega^{\mathbf{2}} = -E_{\mathbf{3}} \cdot DE_{\mathbf{1}}/d\tau$, $\Omega^{\mathbf{3}} = E_{\mathbf{2}} \cdot DE_{\mathbf{1}}/d\tau$. Finally $\varepsilon_{\alpha\beta}{}^{\gamma}$ is the 3-dimensional Levi-Civita symbol, and $R^{\mathbf{i}}{}_\alpha$ denotes the components of the rotation matrix.

Figure 1 shows an example when the spinning body moves on spherical-like orbit in the Kerr spacetime. From left to right the spin of the body is increasing but the other initial conditions are unchanged. The first row represents the orbit of the spinning body while the second and the third rows give the evolutions of $\Omega^{\beta}_{(prec)}$ on different timescales. The first column corresponds to Fig. 3 in Ref. [7], in this case the right hand sides of the MPD equations can be neglected. As the second and the third columns show, the spin and curvature corrections on the right hand sides of the MPD equations become important for higher spins and an amplitude modulation in $\Omega^{\beta}_{(prec)}$ occurs during the evolution.

We note that the static observers only exist outside of the ergosphere of the central rotating black hole. Thus the above description cannot be applied when the spinning body pass over the ergosphere. We introduce another comoving frame which is obtained by boost transformations from the frames of the zero angular momentum observers (ZAMOs). Outside of the ergosphere this comoving frame is related to the previous one by an instantaneous spatial rotation in the rest space of the moving body but it can also be used within the ergosphere. We present the description of the spin evolution in the ZAMO related comoving frame and give numerical results for zoom-whirl orbits passing over the ergosphere. Outside of the ergosphere the spin evolutions in the two comoving frame are compared. The description of the dynamics with TD SSC is also considered.

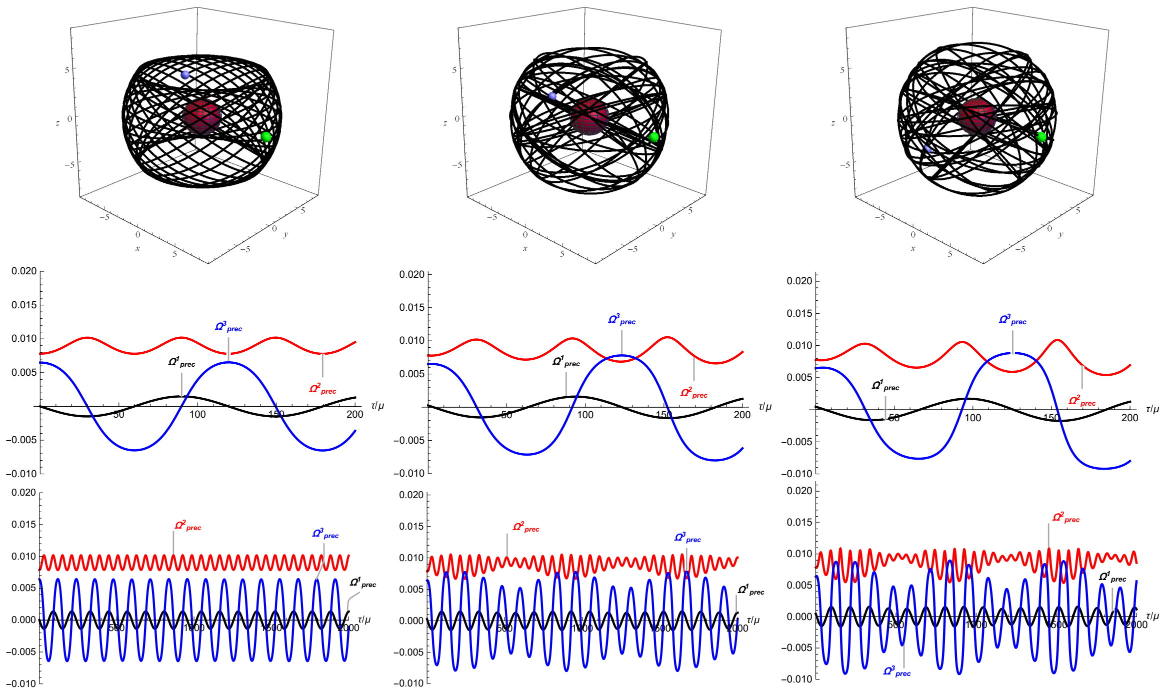

FIG. 1: Spherical orbits around the Kerr black hole with $a = 0.5\mu$ for $(m\mu)^{-1}s = 0.0109$, 0.5454 and 0.9446. The spin magnitude increases from left to right. The first row shows the orbits for increasing spin magnitude. The initial place of the body is $r(0) = 8\mu$, $\theta(0) = \pi/2$ and $\phi(0) = 0$ denoted by the green ball on the orbit. The final place of the spinning body is represented by a blue ball. The ergosphere of the central black hole is marked by red. The second and third rows (under the corresponding orbits) give the evolutions of $\Omega^{\beta}_{(prec)}$ on shorter and longer timescales, respectively.

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