Lorentz–Minkowski zero mass

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In his paper "ÉLECTRICITÉ – Sur la dynamique de l'electron" from June 5th, 1905, Henri Poincaré defined the Lorentz transformations and stated that these form a group [1]. This was a few years after the fundamental papers by H. A. Lorentz [2], W. Voigt [3] and J. J. Larmor [4] and almost at the same time as A. Einstein's paper [5]. A year later, H. Poincare formulated the principle of relativity, introduced the concept of the Lorentz group $\mathrm{Lor}_{1,3}$ and postulated that the laws of Nature must be covariant under Lorentz transformations [6]. It was the mathematician Hermann Minkowski who traced and extended Poincaré's lines of thought in 1908, introducing the 4-dimensional space-time continuum and its invariant metric [7]. As the result of the work of two mathematicians – H. Poincaré and H. Minkowski – the Lorentz group $\mathrm{Lor}_{1,3}$ is the relativistic symmetry group of space-time $M_{1,3}$. The worldline of a particle in the space-time manifold is classified by the value of

$$\tau_P \cdot \tau_P,$$

where τ_P is the tangent vector at the point P. In the case of massive particles the tangent vector τ_P is timelike, $\tau_P \cdot \tau_P > 0$. According to E. P. Wigner the internal symmetry of a massive particle is determined by the little subgroup SO_3 of $\mathrm{Lor}_{1,3}$ [8]. The subgroup $\mathrm{SO}_3 \subset \mathrm{Lor}_{1,3}$ of rotations is the maximal simple and compact subgroup of $\mathrm{Lor}_{1,3}$. On the other hand, the worldline of massless particles has a zero tangent vector at each point P, i.e. $\tau_P \tau_P = 0$. According to the same principle, the intrinsic symmetry group for a non-massive particle must be the maximal, non-simple and non-compact subgroup of $\mathrm{Lor}_{1,3}$, i.e. the maximal solvable subgroup or Borel subgroup $\mathrm{Bor}_{1,3} \subset \mathrm{Lor}_{1,3}$. In the dual space the Borel subgroup is associated with the equation

$$\Lambda \mathring{p} = \lambda \mathring{p}, \qquad \mathring{p} = (\varepsilon, 0, 0, 1), \quad \varepsilon = \pm 1 \text{ and } \lambda > 0.$$

It is clear that Wigner's little group E_2 is a subgroup of the Borel subgroup, $E_2 \subset \mathrm{Bor}_{1,3}$. As a solvable group, $\mathrm{Bor}_{1,3}$ has the structure of a semidirect product

$$\mathrm{Bor}_{1,3} = \mathcal{T}_2 \rtimes \mathrm{Tor}_{1,3},$$

where $\mathcal{T}_2$ is the nilpotent normal subgroup of $\mathrm{Bor}_{1,3}$ and $\mathrm{Tor}_{1,3}$ is the maximal torus of $\mathrm{Bor}_{1,3}$ (and $\mathrm{Lor}_{1,3}$). Using the eigenvalue problem of the adjoint representation one obtains a decomposition of the Lie algebra $\mathrm{bor}_{1,3}$ given by

$$\mathrm{bor}_{1,3} = \mathrm{sol}_2(e) \boxplus \mathrm{sol}_2(f).$$

Here $\boxplus$ is the Kronecker sum and

$$\begin{aligned} \mathrm{sol}_2(e) &= \mathbb{R}e \rtimes \mathbb{R}h_+ = \mathbb{R} \rtimes \mathbb{R} = \mathrm{aff}_1, \\ \mathrm{sol}_2(f) &= \mathbb{R}f \rtimes \mathbb{R}h_- = \mathbb{R} \rtimes \mathbb{R} = \mathrm{aff}_1\,. \end{aligned}$$

The elements h, e, f consitute the standard basis of sl_2, $h_\pm = \frac{1}{2}(1 \pm h)$, and $\rtimes$ is the semidirect sum. Therefore, the Borel algebra $\mathrm{bor}_{1,3}$ is the Kronecker sum of two affine algebras. Since sol_2 is solvable, it determines a single eigenfunction. As a consequence of this, $\mathrm{bor}_{1,3}$ determines two eigenfunctions as the two helicity states for the photon. The special case of the photon (the representation $(\frac{1}{2}, \frac{1}{2})$) provides the classification of zero-mass particles. This leads us to Weinberg's ansatz:

1. If a massless particle is equal to it's antiparticle, it is decribed by the irreducible representation (k, k) of the proper Lorentz group.
2. If a massless particle is not equal to it's antiparticle, the particle is described by the irreducible representation $(k, 0)$ of the proper Lorentz group, and the antiparticle is described by the irreducible representation $(0, k)$ of the proper Lorentz group.

Therefore, for the vector case $(\frac{1}{2}, \frac{1}{2})$ (the photon) there are two helicity states, while for the repesentation $(\frac{1}{2}, 0)$ (the Pauli neutrino) there is only one helicity state. Accordingly, for the representation $(1, 1)$ (a particle of helicity 2) there are four helicity states, two of them being right-handed and two left-handed.

It is useful to note that the classification of massless particles given by Weinberg's ansatz is determined by the algebraic structure (the Borel subgroup) of the proper Lorentz group, i.e. by the symmetry group of Minkowski's space-time.

To conclude, there are two schemes of equal value resulting from the proper Lorentz group:

- the massive case $p^2 = m^2 > 0$, $p \sim \mathring{p} = (m, \vec{0}) \in M_{1,3}$ where $\mathrm{Lor}_{1,3}$ contains the stability or little group SO_3 of the point $\mathring{p}$, the maximal compact simple subroup homeomorphic to the real projective space $P^3(\mathbb{R})$. $\mathcal{Y}_3 = \mathrm{Lor}_{1,3} / \mathrm{SO}_3$ describes a noncompact hyperboloid;
- the massless case $k^2 = 0$, $k \sim \mathring{k} = (1, 0, 0, 1) \in M_{1,3}$ where $\mathrm{Lor}_{1,3}$ contains the little group $\mathrm{Bor}_{1,3}$ of k or the stabilizer of the straight line $\mathbb{R}_+ k$, the maximal noncompact solvable subgroup. $\mathrm{Lor}_{1,3} / \mathrm{Bor}_{1,3} = \mathrm{SO}_3 / \mathrm{SO}_2$ is homeomorphic to the projective variety $P^1(\mathbb{C}) = S^2$.

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